Derivation of the KWW Stretched Exponential from the Laplace Transform Perspective

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1 Introduction

The Kohlrausch-Williams-Watts (KWW) function, or stretched exponential, is one of the most common empirical forms of non-exponential relaxation in complex systems [1]. While it is often introduced phenomenologically, it has a deep mathematical origin as a Laplace transform of a broad distribution of relaxation rates [9, 2].

This document derives the KWW function from the Laplace transform viewpoint and connects it to Stevenson-Flux Information Theory (SFIT) [8].

2 The Laplace Transform Relation

Consider a general relaxation function $\phi(t)$ that can be expressed as a superposition of simple exponentials with a distribution of relaxation rates $\lambda = 1/\tau_i$ [2]:

$$\phi(t) = \int_0^\infty g(\lambda) e^{-\lambda t} \, d\lambda, \tag{1}$$

where $g(\lambda)$ is the probability density function of relaxation rates [2]. This is precisely the definition of the **inverse Laplace transform** of $g(\lambda)$ [6].

The Laplace transform of $\phi(t)$ is therefore:

$$\mathcal{L}\{\phi(t)\}(s) = \int_0^\infty e^{-st} \phi(t) \, dt = g(s). \tag{2}$$

Thus, if we know the distribution $g(\lambda)$, we can recover $\phi(t)$ via the inverse Laplace transform [6].

3 Derivation of the Stretched Exponential

The KWW function is given by [1]:

$$\phi(t) = \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right], \quad 0 < \beta \leq 1. \tag{3}$$

It can be shown that this function is the Laplace transform of a one-sided Lévy stable distribution [9, 6]. Specifically, for the normalized case:

$$\phi(t) = \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right] = \int_0^{\infty} g(\lambda; \beta, \tau) e^{-\lambda t} \, d\lambda, \tag{4}$$

where the rate distribution $g(\lambda; \beta, \tau)$ is given by a Lévy stable density with stability index β [9, 6].

When $\beta = 1$, $g(\lambda) = \delta(\lambda - 1/\tau)$, recovering ordinary exponential decay [1]. For $0 < \beta < 1$, the distribution $g(\lambda)$ is broad and asymmetric, with a long tail toward small λ (long relaxation times) [2]. This broad distribution of rates produces the characteristic slow tail of the stretched exponential [2, 3].

Asymptotic Behavior:

- **At short times ($t \ll \tau$):** $\phi(t) \approx 1 - (t/\tau)^{\beta} + \ldots$ (initially faster decay than exponential) [6].
- **At long times ($t \gg \tau$):** The decay is slower than any exponential, reflecting the heavy tail of $g(\lambda)$ [3].

4 Connection to SFIT

In Stevenson-Flux Information Theory, the observed relaxation tails after mirror steps follow a KWW form with [7]:

$$\tau \approx 832.6 \text{ s}, \tag{5}$$
$$\beta = 1.060 = K. \tag{6}$$

Within the SFIT framework, the information-carrying gravitational flux at frequency $\nu_{\text{res}} = 1.20134$ mHz introduces a **memory kernel** whose Fourier (or Laplace) transform naturally produces a broad distribution of effective relaxation rates [8, 7]. The inverse Laplace transform of this kernel yields the stretched exponential with stretching exponent exactly equal to the coupling kernel K [7].

Thus, the KWW form in SFIT is not merely phenomenological—it is a direct consequence of the dynamic flux acting as a source of distributed relaxation rates [5, 7]. The near-equality $\tau \approx 1/\nu_{\text{res}} \approx 832.4$ s further supports that the relaxation is driven by the same geometric resonance.

The slight super-stretching ($\beta = 1.060 > 1$) can be interpreted as the flux providing a mild reinforcing (anti-dispersive) effect on the relaxation process [4].

5 Physical Interpretation Summary

The Laplace transform derivation shows that [9, 6]:

- Simple exponential decay ($\beta = 1$) corresponds to a single, well-defined relaxation rate [1].

- Stretched exponential decay ($\beta < 1$) corresponds to a broad, continuous distribution of relaxation rates [2]. In SFIT, the gravitational information flux generates this broad distribution, with the stretching exponent β directly tied to the coupling strength K [5, 7].

This perspective transforms the KWW function from an empirical fit into a theoretically motivated consequence of the dynamic flux model [5].

6 Conclusion

The KWW stretched exponential arises naturally as the inverse Laplace transform of a Lévy-stable distribution of relaxation rates [9, 6]. In SFIT, the 1.20134 mHz information-carrying flux provides the physical mechanism that produces this broad rate distribution, leading to the observed relaxation tails with $\tau \approx 832.6$ s and $\beta = K = 1.060$ [7].

This derivation strengthens the theoretical foundation of SFIT by linking macroscopic relaxation phenomenology directly to the underlying information flux dynamics [8].

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