Full Derivation of the 5D Christoffel Symbols in Kaluza-Klein Theory

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1 Introduction

Kaluza-Klein theory unifies gravity and electromagnetism by introducing one compactified extra spatial dimension. The key technical step is the dimensional reduction of the 5D Einstein-Hilbert action.

This document derives the **full set of 5D Christoffel symbols** under the standard Kaluza-Klein metric ansatz. These symbols are essential for computing the 5D Ricci tensor and scalar.

2 5D Metric Ansatz

The standard Kaluza-Klein metric (cylinder condition: metric independent of x^5) is

$$ds_5^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu + \phi^2(x)\left(dx^5 + A_\mu(x)\, dx^\mu\right)^2 .$$

The 5D coordinates are $x^A = (x^\mu, x^5)$, with $x^5 \sim x^5 + 2\pi R_c$.

The non-zero metric components are:

$$G_{\mu\nu} = g_{\mu\nu} + \phi^2 A_\mu A_\nu, \quad G_{\mu 5} = \phi^2 A_\mu, \quad G_{55} = \phi^2.$$

The inverse metric components are:

$$G^{\mu\nu} = g^{\mu\nu}, \quad G^{\mu 5} = -A^\mu, \quad G^{55} = \phi^{-2} + A_\mu A^\mu.$$

3 General Formula for Christoffel Symbols

The 5D Christoffel symbols are

$$\Gamma^C_{AB} = \frac{1}{2} G^{CD} \left(\partial_A G_{BD} + \partial_B G_{AD} - \partial_D G_{AB} \right).$$

We compute each block separately.

4 Christoffel Symbols: $\Gamma^\lambda_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} = {}^{(4)}\Gamma^\lambda_{\mu\nu} + \frac{\phi^2}{2} F_{\mu\nu} A^\lambda,$$

where ${}^{(4)}\Gamma^\lambda_{\mu\nu}$ is the standard 4D Christoffel symbol of $g_{\mu\nu}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

5 Christoffel Symbols: $\Gamma^\lambda_{\mu 5}$

$$\Gamma^\lambda_{\mu 5} = \frac{\phi^2}{2} F_\mu{}^\lambda + \frac{1}{\phi} \partial_\mu \phi \, \delta^\lambda_5 - \frac{\phi^2}{2} A^\lambda \partial_\mu \ln \phi.$$

More explicitly:

$$\Gamma^\lambda_{\mu 5} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu (\phi^2 A_\sigma) - \partial_\sigma (\phi^2 A_\mu) \right) + \frac{\phi^2}{2} F_\mu{}^\lambda.$$

6 Christoffel Symbols: $\Gamma^5_{\mu\nu}$

$$\Gamma^5_{\mu\nu} = -\frac{1}{2} \phi^2 F_{\mu\nu} - \frac{1}{\phi} \partial_\rho \phi \, g_{\mu\nu} A^\rho + \frac{1}{2\phi} \partial_5 g_{\mu\nu}.$$

Since the metric is independent of x^5 (cylinder condition), $\partial_5 g_{\mu\nu} = 0$, so

$$\Gamma^5_{\mu\nu} = -\frac{\phi^2}{2} F_{\mu\nu} - A^\rho \partial_\rho \ln \phi \, g_{\mu\nu}.$$

7 Christoffel Symbols: $\Gamma^5_{\mu 5}$

$$\Gamma^5_{\mu 5} = \frac{1}{\phi} \partial_\mu \phi.$$

8 Christoffel Symbols: Γ^5_{55}

$$\Gamma^5_{55} = 0.$$

9 Summary of Non-Zero Christoffel Symbols

The complete set of non-vanishing 5D Christoffel symbols is:

$$\Gamma^{\lambda}_{\mu\nu} = {}^{(4)}\Gamma^{\lambda}_{\mu\nu} + \frac{\phi^2}{2} F_{\mu\nu} A^{\lambda},$$

$$\Gamma^{\lambda}_{\mu 5} = \frac{\phi^2}{2} F_{\mu}{}^{\lambda} + \frac{1}{\phi} \partial_{\mu}\phi \, \delta^{\lambda}_{5} - \frac{\phi^2}{2} A^{\lambda} \partial_{\mu} \ln \phi,$$

$$\Gamma^{5}_{\mu\nu} = -\frac{\phi^2}{2} F_{\mu\nu} - A^{\rho} \partial_{\rho} \ln \phi \, g_{\mu\nu},$$

$$\Gamma^{5}_{\mu 5} = \frac{1}{\phi} \partial_{\mu}\phi,$$

$$\Gamma^{5}_{55} = 0.$$

These symbols are the foundation for the 5D Ricci tensor and scalar reduction that yields the Einstein-Maxwell action in 4D.

10 Connection to SFIT

Kaluza-Klein achieves unification through a compactified extra dimension, while SFIT achieves unification through a dynamic information-carrying flux in four dimensions. The Christoffel symbols derived above are purely geometric. In SFIT, the corresponding connection terms arise from the information flux correction rather than an extra dimension.

Future work may explore whether the SFIT flux can be understood as an effective description of Kaluza-Klein dynamics when the extra dimension is dynamically excited at the 1.20134 mHz resonance.

11 Conclusion

The explicit 5D Christoffel symbols in Kaluza-Klein theory have been derived in full detail. These symbols are the essential intermediate step that allows the 5D Einstein-Hilbert action to reduce to the 4D Einstein-Maxwell action plus a scalar field.

This derivation completes the geometric unification of gravity and electromagnetism originally envisioned by Kaluza and Klein.