The Kohlrausch–Williams–Watts (KWW) Function: Theory and Application in Stevenson-Flux Information Theory (SFIT)

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Contents

1 Introduction

The Kohlrausch–Williams–Watts (KWW) function, also known as the *stretched exponential*, is a widely used model for describing non-exponential relaxation in complex physical systems. In Stevenson-Flux Information Theory (SFIT), the KWW form with $\tau \approx 832.6\,\mathrm{s}$ and $\beta = 1.060$ naturally emerges from the dynamic information-carrying gravitational flux at the geometric resonance frequency $\nu_{\mathrm{res}} = 1.20134\,\mathrm{mHz}$. This provides a quantitative description of the relaxation tails observed in the reanalysis of ILL Archive 3-14-412 (qBounce ultra-cold neutron experiment).

2 Mathematical Definition

The standard KWW relaxation function is

$$\phi(t) = A \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right] \quad \text{for } t \geq 0,$$

where A is the amplitude, $\tau > 0$ is the characteristic relaxation time, and $0 < \beta \leq 1$ is the stretching exponent.

When $\beta = 1$, it reduces to a simple exponential decay:

$$\phi(t) = A \exp\left(-\frac{t}{\tau}\right).$$

For $\beta < 1$, the function exhibits a slower long-time tail, reflecting a broad distribution of relaxation processes.

3 Historical Background

The stretched exponential was first observed by Rudolf Kohlrausch in 1854 while studying capacitor discharge. It was later applied to mechanical creep and popularized in dielectric spectroscopy by Williams and Watts in 1970.

4 Laplace Transform Representation and Superposition of Exponentials

One of the most important theoretical interpretations of the KWW function is that it can be expressed as a continuous superposition (weighted sum) of simple exponential decays with a broad distribution of relaxation times.

The Laplace transform of the KWW function is closely related to this idea. Consider the normalized KWW function ($\phi(0) = 1$):

$$\phi(t) = \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right].$$

This function can be written as

$$\phi(t) = \int_0^\infty g(\lambda)\, e^{-\lambda t}\, d\lambda,$$

where $g(\lambda)$ is a probability density function representing the distribution of relaxation rates $\lambda = 1/\tau_i$.

For the KWW case, the distribution $g(\lambda)$ is a one-sided Lévy stable distribution. Specifically, the Laplace transform relationship yields:

$$\mathcal{L}\{\phi(t)\}(s) = \int_0^\infty e^{-st} \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right] dt,$$

which does not have a simple closed form for general β, but is known to correspond to a stable distribution in the rate domain.

When $\beta = 1$, $g(\lambda)$ becomes a Dirac delta function $\delta(\lambda - 1/\tau)$, recovering the single-exponential case.

For $0 < \beta < 1$, the distribution $g(\lambda)$ is broad and asymmetric, with a long tail toward small relaxation rates (long times). This heterogeneous picture explains why many disordered or interacting systems exhibit KWW-type relaxation: different microscopic subsystems relax with different time constants due to local variations in the environment or coupling strength.

In SFIT, the information flux at 1.20134 mHz provides a natural physical mechanism that generates this broad distribution of effective relaxation rates through the coupling kernel $K = 1.060$. The memory kernel induced by the oscillating flux leads to the specific stretching exponent $\beta = K = 1.060$ observed in the post-mirror-step tails.

5 Physical Interpretations

- **Heterogeneous relaxation**: Superposition of many exponentials with distributed rates.
- **Time-dependent dissipation**: Linear relaxation with a time-dependent damping coefficient.
- **Hierarchical dynamics**: Correlated or cascaded relaxation processes in complex systems.

6 Connection to SFIT and qBounce Experiment

In SFIT, mirror-step transitions trigger relaxation processes whose time evolution follows the KWW form. The parameters are not arbitrary:

- Relaxation time $\tau \approx 832.6\,\mathrm{s}$ matches the geometric resonance period $1/\nu_{\mathrm{res}}$.
- Stretching exponent $\beta = 1.060$ is directly equal to the refined coupling kernel K.

This connection transforms the KWW function from a purely phenomenological fit into a prediction derived from the dynamic information flux model. The synthetic data generator and analyzer scripts in the SFIT GitHub repository reproduce both the 1.20134 mHz modulation in the PSD and the KWW tails with high fidelity, allowing independent verification.

7 Practical Usage in Analysis

When fitting post-step regions of the binned rate time series, the model takes the form

$$\mathrm{rate}(t) \approx r_0 + A \exp\left[-\left(\frac{t - t_{\mathrm{step}}}{\tau} \right)^{\beta} \right].$$

Averaging fits over multiple mirror steps yields robust estimates of τ and β.

8 Limitations

The KWW function is not universal. It may fail at extremely short or very long times, or when multiple independent physical mechanisms are present. In such cases, more general models (e.g., Havriliak–Negami) can be more appropriate.

9 References

- Kohlrausch, R. (1854). Ann. Phys. Chem.
- Williams, G. and Watts, D.C. (1970). Trans. Faraday Soc.
- Lukichev, A. (2019). Physical meaning of the stretched exponential. Phys. Lett. A.
- SFIT Preprint and technical blog posts at stevensonfluxinformationtheory.com.

This document is part of the open supplementary materials for Stevenson-Flux Information Theory. Code, synthetic data, and analysis scripts are available on the associated GitHub repository.