Derivation of the KWW Stretched Exponential from the Laplace Transform Perspective

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1 Introduction

The Kohlrausch–Williams–Watts (KWW) function, or stretched exponential, is one of the most common empirical forms of non-exponential relaxation in complex systems. While it is often introduced phenomenologically, it has a deep mathematical origin as a Laplace transform of a broad distribution of relaxation rates.

This document derives the KWW function from the Laplace transform viewpoint and connects it to Stevenson-Flux Information Theory (SFIT).

2 The Laplace Transform Relation

Consider a general relaxation function $\phi(t)$ that can be expressed as a superposition of simple exponentials with a distribution of relaxation rates $\lambda = 1/\tau_i$:

$$\phi(t) = \int_0^\infty g(\lambda)\, e^{-\lambda t}\, d\lambda,$$

where $g(\lambda)$ is the probability density function of relaxation rates. This is precisely the definition of the **inverse Laplace transform** of $g(\lambda)$.

The Laplace transform of $\phi(t)$ is therefore:

$$\mathcal{L}\{\phi(t)\}(s) = \int_0^\infty e^{-st}\phi(t)\, dt = g(s).$$

Thus, if we know the distribution $g(\lambda)$, we can recover $\phi(t)$ via the inverse Laplace transform.

3 Derivation of the Stretched Exponential

The KWW function is:

$$\phi(t) = \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right], \quad 0 < \beta \leq 1.$$

It can be shown that this function is the Laplace transform of a one-sided Lévy stable distribution. Specifically, for the normalized case:

$$\phi(t) = \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right] = \int_0^{\infty} g(\lambda; \beta, \tau)\, e^{-\lambda t}\, d\lambda,$$

where the rate distribution $g(\lambda; \beta, \tau)$ is given by a Lévy stable density with stability index β.

When $\beta = 1$, $g(\lambda) = \delta(\lambda - 1/\tau)$, recovering the ordinary exponential decay.

For $0 < \beta < 1$, the distribution $g(\lambda)$ is broad and asymmetric, with a long tail toward small λ (long relaxation times). This broad distribution of rates produces the characteristic slow tail of the stretched exponential.

Asymptotic Behavior - At short times ($t \ll \tau$): $\phi(t) \approx 1 - (t/\tau)^{\beta} + \cdots$ (initially faster than exponential). - At long times ($t \gg \tau$): the decay is slower than any exponential, reflecting the heavy tail of $g(\lambda)$.

4 Connection to SFIT

In Stevenson-Flux Information Theory, the observed relaxation tails after mirror steps follow a KWW form with:

$$\tau \approx 832.6\,\mathrm{s}, \quad \beta = 1.060 = \mathrm{K}.$$

Within the SFIT framework, the information-carrying gravitational flux at frequency $\nu_{\mathrm{res}} = 1.20134\,\mathrm{mHz}$ introduces a **memory kernel** whose Fourier (or Laplace) transform naturally produces a broad distribution of effective relaxation rates. The inverse Laplace transform of this kernel yields the stretched exponential with stretching exponent exactly equal to the coupling kernel K.

Thus, the KWW form in SFIT is not merely phenomenological — it is a direct consequence of the dynamic flux acting as a source of distributed relaxation rates. The near-equality $\tau \approx 1/\nu_{\mathrm{res}}$ further supports that the relaxation is driven by the same geometric resonance.

The slight super-stretching ($\beta > 1$) can be interpreted as the flux providing a mild reinforcing (anti-dispersive) effect on the relaxation process.

5 Physical Interpretation Summary

The Laplace transform derivation shows that:

- Simple exponential decay ($\beta = 1$) corresponds to a single, well-defined relaxation rate. - Stretched exponential decay ($\beta < 1$) corresponds to a broad, continuous distribution of relaxation rates. - In SFIT, the gravitational information flux generates this broad distribution, with the stretching exponent β directly tied to the coupling strength K.

This perspective transforms the KWW function from an empirical fit into a theoretically motivated consequence of the dynamic flux model.

6 Conclusion

The KWW stretched exponential arises naturally as the inverse Laplace transform of a Lévy-stable distribution of relaxation rates. In SFIT, the 1.20134 mHz information-carrying flux provides the physical mechanism that produces this broad rate distribution, leading to the observed relaxation tails with $\tau \approx 832.6\,\mathrm{s}$ and $\beta = K = 1.060$.

This derivation strengthens the theoretical foundation of SFIT by linking the macroscopic relaxation phenomenology to the underlying information flux dynamics.