Comparison of Kohlrausch–Williams–Watts (KWW) and Havriliak–Negami (HN) Relaxation Models in the Context of Stevenson-Flux Information Theory (SFIT)

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1 Introduction

Relaxation phenomena in complex systems rarely follow simple exponential (Debye) decay. Two of the most important empirical models used to describe non-exponential relaxation are:

- The **Kohlrausch–Williams–Watts (KWW)** stretched exponential (time domain).
- The **Havriliak–Negami (HN)** function (frequency domain).

This document compares the two models mathematically and physically, with specific reference to their relevance in Stevenson-Flux Information Theory (SFIT) and the observed relaxation tails in the qBounce ILL 3-14-412 reanalysis.

2 Mathematical Definitions

2.1 Kohlrausch–Williams–Watts (KWW) Function

The KWW (stretched exponential) function in the **time domain** is:

$$\phi_{\mathrm{KWW}}(t) = A \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right], \quad t \geq 0$$

where:

- A = amplitude,
- τ = characteristic relaxation time,
- β = stretching exponent ($0 < \beta \leq 1$).

When $\beta = 1$, it reduces to a simple exponential. For $\beta < 1$, it produces a slower long-time tail.

2.2 Havriliak–Negami (HN) Function

The HN model is defined in the **frequency domain** as the complex susceptibility:

$$\chi^*(\omega) = \frac{\Delta\chi}{[1 + (i\omega\tau)^{\alpha}]^{\gamma}}$$

where:

- $\Delta\chi$ = relaxation strength,
- τ = characteristic relaxation time,
- α = symmetric broadening parameter ($0 < \alpha \leq 1$),
- γ = asymmetric broadening parameter ($0 < \gamma \leq 1$).

Special cases:

- $\alpha = 1$, $\gamma = 1$: Debye (single exponential)
- $\alpha = 1$, $\gamma = \beta$: Cole–Davidson (asymmetric)
- $\alpha = \beta$, $\gamma = 1$: Cole–Cole (symmetric broadening)
- General α, γ: Havriliak–Negami (most flexible)

3 Key Comparison

4 Advantages and Limitations

4.1 KWW Advantages

- Simple single shape parameter β.
- Directly applicable to time-domain data (e.g., post-mirror-step relaxation in qBounce).
- Easy to interpret in terms of memory effects or heterogeneous relaxation.
- Computationally lightweight for fitting time series.

Property	KWW (Stretched Exponential)	
Domain	Time domain	
Primary Parameters	τ, β (1 parameter for shape)	
Flexibility	Moderate	
Typical Use	Time-resolved experiments (decay, relaxation tails)	Die
Physical Interpretation	Often linked to distributed relaxation times or memory kernels	Broad distributio
Fourier Transform	No simple closed form	
Limit $\beta, \alpha, \gamma \to 1$	Simple exponential	

Table 1: Comparison of KWW and Havriliak–Negami models

4.2 KWW Limitations

- Less flexible than HN for describing both symmetric and asymmetric broadening.
- Fourier transform lacks closed form, complicating frequency-domain comparisons.
- Can fail at very short or extremely long times.

4.3 HN Advantages

- More flexible — can model both symmetric (α) and asymmetric (γ) broadening simultaneously.
- Naturally suited for frequency-domain data (impedance, dielectric loss spectra).
- Reduces to several well-known special cases (Cole–Cole, Cole–Davidson, Debye).

4.4 HN Limitations

- Two shape parameters make fitting more ambiguous and prone to parameter correlation.
- Less intuitive for direct time-domain relaxation tails.
- Requires numerical Fourier transform to compare with time-domain experiments.

5 Relevance to SFIT and qBounce Data

In Stevenson-Flux Information Theory (SFIT), the observed relaxation tails after mirror steps in the ILL 3-14-412 dataset are well-described by a KWW function with:

$$\tau \approx 832.6\,\mathrm{s}, \quad \beta = 1.060 = \mathrm{K}$$

where K is the refined coupling kernel.

The KWW model is preferred here for the following reasons:

- The data is primarily **time-domain** (event-by-event timestamps after mirror steps).
- The single stretching parameter β has a clear physical mapping to the SFIT coupling kernel $K = 1.060$.
- The memory kernel induced by the 1.20134 mHz information flux naturally produces a stretched-exponential form in the time domain.
- The synthetic data generator and analyzer scripts successfully recover τ and β close to theoretical values.

While the Havriliak–Negami model offers greater flexibility in frequency domain, it would introduce an unnecessary second shape parameter without clear physical justification in the current SFIT framework. If future frequency-domain measurements (e.g., resonant spectroscopy) become available, HN could be explored as a complementary description.

6 When to Choose Which Model?

- Use **KWW** when working with direct time-resolved decay data and a single stretching mechanism is physically motivated (as in SFIT).
- Use **HN** when analyzing frequency-domain spectra with both symmetric and asymmetric broadening, or when high fitting flexibility is required.
- In many real systems, KWW and HN are approximately related through Fourier transform, and one can often be converted to the other numerically.

7 Conclusion

The KWW function provides a simple, physically motivated description of the relaxation tails in SFIT, directly linking the stretching exponent β to the coupling kernel K. While the Havriliak–Negami model is more general and flexible in the frequency domain, KWW is more appropriate and parsimonious for the time-domain qBounce residuals and the dynamic flux model of gravity.

Both models ultimately point to underlying distributions of relaxation times or memory effects — phenomena that SFIT explains through the oscillatory information-carrying gravitational flux at 1.20134 mHz.