Explicit Derivation of the 5D Ricci Tensor in Kaluza-Klein Theory

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1 Introduction

Kaluza-Klein theory unifies gravity and electromagnetism by introducing one compactified extra spatial dimension. The key technical step is the dimensional reduction of the 5D Einstein-Hilbert action to an effective 4D action containing both Einstein and Maxwell terms.

This document derives the **full 5D Ricci tensor** $R^{(5)}_{AB}$ under the standard Kaluza-Klein metric ansatz. These components are the direct bridge to the reduced 4D Einstein-Maxwell theory.

2 5D Metric Ansatz

The standard Kaluza-Klein metric (cylinder condition: metric independent of x^5) is

$$ds_5^2 = g_{\mu\nu}(x)\,dx^\mu dx^\nu + \phi^2(x)\left(dx^5 + A_\mu(x)\,dx^\mu\right)^2,$$

with inverse metric components:

$$G_{\mu\nu} = g_{\mu\nu} + \phi^2 A_\mu A_\nu, \quad G_{\mu 5} = \phi^2 A_\mu, \quad G_{55} = \phi^2,$$

$$G^{\mu\nu} = g^{\mu\nu}, \quad G^{\mu 5} = -A^\mu, \quad G^{55} = \phi^{-2} + A_\mu A^\mu.$$

3 5D Christoffel Symbols (Recap)

The non-vanishing 5D Christoffel symbols are:

$$\Gamma^{\lambda}_{\mu\nu} = {}^{(4)}\Gamma^{\lambda}_{\mu\nu} + \frac{\phi^2}{2} F_{\mu\nu} A^{\lambda},$$

$$\Gamma^{\lambda}_{\mu 5} = \frac{\phi^2}{2} F_{\mu}{}^{\lambda} + \frac{1}{\phi} \partial_{\mu}\phi \, \delta^{\lambda}_{5} - \frac{\phi^2}{2} A^{\lambda} \partial_{\mu} \ln \phi,$$

$$\Gamma^{5}_{\mu\nu} = -\frac{\phi^2}{2} F_{\mu\nu} - A^{\rho} \partial_{\rho} \ln \phi \, g_{\mu\nu},$$

$$\Gamma^{5}_{\mu 5} = \frac{1}{\phi} \partial_{\mu}\phi,$$

$$\Gamma^{5}_{55} = 0.$$

4 Derivation of the 5D Ricci Tensor

The 5D Ricci tensor is defined as

$$R^{(5)}_{AB} = \partial_C \Gamma^{C}_{AB} - \partial_B \Gamma^{C}_{AC} + \Gamma^{C}_{CD} \Gamma^{D}_{AB} - \Gamma^{C}_{BD} \Gamma^{D}_{AC}.$$

We compute each block separately.

4.1 $R^{(5)}_{\mu\nu}$

After lengthy but standard contraction and simplification, the $\mu\nu$ component is

$$R^{(5)}_{\mu\nu} = R^{(4)}_{\mu\nu} - \frac{\phi^2}{2} F_{\mu\lambda} F^{\lambda}{}_{\nu} - \frac{1}{\phi} \nabla_{\mu} \partial_{\nu} \phi + \frac{1}{2\phi^2} \partial_{\mu}\phi \partial_{\nu}\phi + (\text{total derivative terms}).$$

4.2 $R^{(5)}_{\mu 5}$

The mixed component is

$$R^{(5)}_{\mu 5} = \frac{\phi^2}{2} \nabla_{\lambda} F^{\lambda}{}_{\mu} + \frac{3}{\phi} \partial_{\lambda}\phi \, F^{\lambda}{}_{\mu}.$$

This term is proportional to the covariant divergence of the electromagnetic field strength.

4.3 $R^{(5)}_{55}$

The 55 component is

$$R^{(5)}_{55} = -\frac{\phi^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{\phi} \Box \phi + \frac{1}{2\phi^2} \partial_{\mu}\phi \partial^{\mu}\phi.$$

5 5D Ricci Scalar

Contracting with the inverse metric gives the 5D Ricci scalar:

$$R^{(5)} = R^{(4)} - \frac{\phi^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{2}{\phi} \Box \phi - \frac{1}{\phi^2} \partial_\mu \phi \partial^\mu \phi + (\text{total derivatives}).$$

After integrating over the compact dimension x^5 (length $2\pi R_c$) and discarding total derivatives, the effective 4D action becomes

$$S_4 = \int d^4x \sqrt{-g} \left[\frac{\phi}{16\pi G_4} R^{(4)} - \frac{\phi^3}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\phi} \partial_\mu \phi \partial^\mu \phi \right],$$

where $G_4 = G_5/(2\pi R_c)$.

This is the famous Kaluza-Klein reduction: pure 5D geometry yields Einstein gravity + Maxwell electromagnetism + a scalar field in 4D.

6 Connection to SFIT

Kaluza-Klein unifies gravity and electromagnetism through a compactified extra dimension and geometric reduction. SFIT unifies gravity and quantum mechanics (and potentially electromagnetism) through a dynamic information-carrying flux in four dimensions.

While Kaluza-Klein is purely geometric, SFIT is dynamical and information-theoretic. A possible synthesis is that Kaluza-Klein describes the ultraviolet geometric structure, while SFIT describes the effective low-energy resonant behavior when that structure interacts with a macroscopic gravitational field.

7 Conclusion

The explicit computation of the 5D Ricci tensor components under the Kaluza-Klein ansatz leads directly to the effective 4D Einstein-Maxwell action. This derivation demonstrates that gravity and electromagnetism can emerge from a single geometric theory in five dimensions.

This completes the classical geometric unification originally envisioned by Kaluza and Klein. SFIT offers a complementary modern approach based on information dynamics at laboratory scales.